Multiple-Description Vector Quantization using Translated Lattices with Local Optimization

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***Abstract*— Multiple-description coding is a joint source- and channel coding technique suitable for real-time multimedia transmission over erasure channels. This work improves the previous methods of multiple-description vector quantization using lattice structured codebooks by introducing translated lattices in the single-description codebooks. The quantizer can easily adapt to the current channel condition, using the locally optimized combined-description codebooks, assuming that channel statistics are available at the encoder. Compared to previous methods, the central distortion is greatly reduced for noisy channels, without a significant effect on complexity.**

I. Introduction

Using multiple-description coding (MDC), a source is encoded in two or more distinct descriptions. It is assumed that the descriptions are transmitted over separate erasure channels, such that the probability of losing one description is independent of losing other descriptions. Any individual description can be decoded with reasonable quality, and the decoding quality will improve with additional received descriptions. The application of MDC can be found in real-time multimedia transmissions over erasure channels, in particular, packet switched networks such as the Internet. As the channel statistics can be made available at the encoder, using a feedback channel through the Real-Time Control Protocol (RTCP), the MDC encoder can optimize the descriptions for the instantaneous channel condition.

The asymptotic performance of MDC, as dimension and rate approach infinity, has been studied in terms of the MDC rate-distortion bound in classical papers [1][2]. The first practical multiple-description scalar quantizer (MDSQ) design using two descriptions was developed more recently in [3][4]. The training method in MDSQ has been extended in [5] to an MD vector quantizer (MDVQ) and to the case of more than two descriptions. These trained quantizers have optimal performance such that the average total distortion is minimized for given channel statistics. However, there is one major drawback associated with the unconstrained methods: high computational complexity in codebook training and, more importantly, encoding. In the case of nonstationary channels, optimal performance requires multiple codebooks, each optimized for a certain channel condition, and selected according to the current channel condition. Training and storage of multiple codebooks becomes a practical problem.

One solution to the computational problem is to use structured codebooks, and the design of an MD lattice VQ (MDLVQ) has been considered by Servetto, Vaishampayan and Sloane (SVS-MDLVQ) in [6][7], where a lattice combined-description codebook and a sublattice single-description codebook is used. The complexity is greatly reduced using the geometrical structure of the chosen lattice and sublattice. The optimal codebooks for instantaneous channel statistics are determined by the optimal *index* N of the sublattice, which is the number of lattice points within each Voronoi cell of one sublattice. However, as pointed out in [8][9], the SVS-MDLVQ method is inherently optimal only for low erasure probabilities and that the number of available codebooks for higher erasure probabilities is very limited. To address this drawback, a new method was proposed by Kelner, Goyal and Kovačević (KGK-MDLVQ) that modifies the encoder and the local codebook (to a nonlattice codebook) to minimize the average total distortion. Using KGK-MDLVQ, more useful operating points can be made for a large range of erasure probabilities.

In this work, we show that the performance of KGK-MDLVQ can be improved further, based on the observation that KGK-MDLVQ converges to repetition codes for increased erasure probabilities. Repetition codes produce inefficient descriptions for the central decoder, as no improvement is gained when both descriptions are available. We extend the SVS-MDLVQ method to allow translated lattices in the single-description codebooks, and the combined-description codebook is locally trained similarly to KGK-MDLVQ.

Some notations and the theory are introduced in section II. Our construction of MDVQ using translated lattices and local optimization (MDVQ-TLLO) is described in section III and the performance of the method is evaluated in section IV. Section V summarizes and provides some conclusions.

II. Theory

In this section, we introduce the notations and the theory. We first define a k-dimensional lattice Λ as an infinite set of vectors generated by a generating matrix $\mathbf{G}$:

$$\Lambda = \{\mathbf{G}^T\mathbf{u} : \mathbf{u} \in \mathbb{Z}^k\}, \tag{1}$$

where $\mathbf{G}$ is assumed to be a square $k \times k$ matrix with linearly independent row vectors. A translated lattice Λ_i, given lattice Λ and a translation offset $\mathbf{o}_i \in \mathbb{R}^k$ is:

$$\Lambda_i = \{\lambda + \mathbf{o}_i : \forall \lambda \in \Lambda\}. \quad (2)$$

We can then create a set of m translated lattices, $[\Lambda_1, \Lambda_2, ..., \Lambda_m]$ (Λ_1 has the trivial offset $\mathbf{o}_1 = 0$, equivalently $\Lambda_1 = \Lambda$). Each translated lattice is the codebook for a particular description.

Here we only consider the case of two descriptions, $m = 2$, with Λ_1 and Λ_2 as single-description codebooks. For encoding and for decoding when both descriptions arrive, we define the combined-description codebook $\mathcal{C}$ that has the structure:

$$\mathcal{C} = \{\lambda + \mathcal{C}', \forall \lambda \in \Lambda_1\}, \quad (3)$$

where $\mathcal{C}'$ is the local codebook consisting of N codevectors. N is closely related to the sublattice index in [7], which reflects the amount of redundancy in the descriptions[1]. Without losing generality, we assume that $\mathcal{C}'$ lies in the Voronoi cell of the lattice vector $\lambda^o \in \Lambda_1$ at the origin. The definition generalizes [7] in which the combined-description codebook is assumed to be a lattice as well. We denote the quantization cell of a codevector $\mathbf{c} \in \mathcal{C}$ by $\mathcal{V}(\mathbf{c})$, and the Lebesgue measure of the cell by $\mathrm{vol}(\mathbf{c})$.

Let $\mathbf{x} = [x_1, x_2, \ldots, x_k]^T$ be an input vector in the k-dimensional Euclidean space $\mathbb{R}^k$. We define the encoder of MDVQ-TLLO as a two-step procedure $\mathcal{Q}(\mathbf{x}) = \ell(\mathcal{E}(\mathbf{x}))$. The first step is the mapping $\mathcal{E}$ of $\mathbb{R}^k$ onto the combined-description codebook $\mathcal{C}$,

$$\mathcal{E} : \mathbb{R}^k \to \mathcal{C}. \quad (4)$$

The second step is an injective mapping ℓ from $\mathcal{C}$ to $m = 2$ translated-lattice single-description codebooks,

$$\ell : \mathcal{C} \xrightarrow{\text{inj.}} [\Lambda_1, \Lambda_2]. \quad (5)$$

The forward mapping $\ell(\mathbf{c})$ consists of two separate mappings to single-description codebooks, $\ell(\mathbf{c}) = [\ell_1(\mathbf{c}), \ell_2(\mathbf{c})]$.

We assume that the translated lattices are to be entropy coded and transmitted separately over independent erasure channels. For the case of two descriptions, three decoders can be designed. When only one description $i \in \{1, 2\}$ is available, the so-called side decoder $\mathcal{D}_{s_i}$ simply uses the corresponding translated lattice vector to construct the output $\hat{\mathbf{x}}_{s_i}$. When both descriptions are available, the so-called central decoder $\mathcal{D}_c$ uses the inverse mapping function ℓ^{-1} to find the combined-description coded vector $\hat{\mathbf{x}}_c$ of $\mathcal{C}$. The structure of two-description MDVQ-TLLO is shown in figure 1.

We assume the single-letter squared error distortion measure

$$d(\mathbf{x}, \hat{\mathbf{x}}_i) = \frac{1}{k}||\mathbf{x} - \hat{\mathbf{x}}_i||^2, \quad (6)$$

for the decoder output $\hat{\mathbf{x}}_i$ of decoder $i \in \{c, s_1, s_2\}$. The expected distortion for for the side decoder $\mathcal{D}_{\{s_1,s_2\}}$ is referred

[1]Unlike [7], the choice of N mainly affects the descriptions optimized for the zero-erasure channel. While a large local codebook reduces the central distortion, the complexity is increased for increased codebook size.

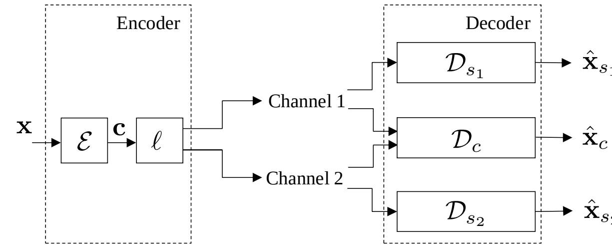

Fig. 1. A block diagram of a multiple-description quantizer.

to as the *side distortion*, denoted as $D_{\{s_1,s_2\}}$, and the *central distortion* for the central decoder $\mathcal{D}_c$, denoted D_c.

The choice of lattice codebooks can be motivated using high-rate theory, where it is assumed that the data can be approximated as uniformly distributed in each cell. Under this assumption, a quantizer using a lattice codebook is optimal if the lattice has the optimal cell shape and codevectors are entropy coded.

Additional motivations for lattice codebooks include that no storage is needed and that the closest lattice point search can be performed efficiently [10][11], which corresponds to very fast encoding for the single-letter squared error distortion measure. While the combined-description codebook $\mathcal{C}$ is not a lattice, we impose a certain structure constraint in section III to reduce the complexity and storage requirement.

A. Distortion analysis

We now study the quantization error of MDVQ-TLLO, making high-rate assumptions. Let $\mathbf{X}$ denote a random input vector with probability density (pdf) $f(\mathbf{x})$. Exploiting that the codevectors in the codebook $\mathcal{C}$ have the same structure around each lattice point λ, the expected distortions are given by:

$$\begin{aligned} D_{i=c,s_1,s_2} &= \int f(\mathbf{x}) d(\mathbf{x}, \hat{\mathbf{x}}_i) d\mathbf{x} \\ &\approx \frac{1}{\mathrm{vol}(\lambda^o)} \int_{\mathcal{W}(\lambda^o)} d(\mathbf{x}, \hat{\mathbf{x}}_i) d\mathbf{x}, \end{aligned} \quad (7)$$

where $\mathcal{W}(\lambda^o)$ is the Voronoi region of λ^o.

Expanding (7) using the squared error distortion (6) gives the central distortion:

$$D_c \approx \frac{1}{\mathrm{vol}(\lambda^o)k} \sum_{\mathbf{c} \in \mathcal{C}'} \int_{\mathbf{x} \in \mathcal{V}(\mathbf{c})} ||\mathbf{x} - \mathbf{c}||^2 d\mathbf{x}, \quad (8)$$

and the side distortion for description j:

$$\begin{aligned} D_{s_j} &\approx \frac{1}{\mathrm{vol}(\lambda^o)k} \sum_{\mathbf{c} \in \mathcal{C}'} \int_{\mathbf{x} \in \mathcal{V}(\mathbf{c})} ||\mathbf{x} - \ell_j(\mathbf{c})||^2 d\mathbf{x} \\ &\approx D_c + \frac{1}{\mathrm{vol}(\lambda^o)k} \sum_{\mathbf{c} \in \mathcal{C}'} \int_{\mathbf{x} \in \mathcal{V}(\mathbf{c})} ||\mathbf{c} - \ell_j(\mathbf{c})||^2 d\mathbf{x}, \end{aligned} \quad (9)$$

where we made the same approximation as [7] in their equation (7).

As shown in MDC rate-distortion theory [2], the central- and side distortion cannot be simultaneously optimal. The central distortion (8) can be minimized by optimizing the local codebook $\mathcal{C}'$. If we optimize (9) only for the mapping function ℓ (i.e., for given $\mathcal{C}'$), then the central distortion is unaffected.

B. Channel optimization

Next, we consider optimization of the two-description coder for channels with known statistics. We assume that the channels have equal statistics, and the probability of losing each description is denoted p. The average side distortion is $D_s = (D_{s_1} + D_{s_2})/2$, and the *average total distortion* $\bar{D}$ can be expressed as a function of p, D_c and D_s:

$$\bar{D} = (1-p)^2 D_c + 2p(1-p)D_s + p^2 D_z, \quad (10)$$

where D_z denotes the distortion when both descriptions are lost (zero-description). For a stationary source and squared error, D_z is essentially the variance of the source. For known p, we aim to optimize the encoder $\mathcal{E}$ and the combined-description codebook $\mathcal{C}$ such that the pair of achieved D_c and D_s minimizes $\bar{D}$, optimal for the channel condition. The optimal encoder and codebook are then defined as:

$$\begin{aligned}\{\mathcal{E}, \mathcal{C}\}_{\text{opt}} &= \arg\min_{\mathcal{E},\mathcal{C}} \ \bar{D} \\ &= \arg\min_{\mathcal{E},\mathcal{C}} \ D_c + \frac{p}{1-p}(D_{s_1} + D_{s_2}), \quad (11)\end{aligned}$$

and we introduce the parameter $\beta = \frac{p}{1-p}$, which can be seen as a Lagrange multiplier for minimization of the central distortion given the average side distortion. For the case $\beta = 0$, we minimize the central distortion only and ignore the performance of the side decoders. When β approaches infinity (p goes to 1), only side distortions are minimized. By optimizing for a particular value of β, an optimal operating point for the two-description coder at erasure probability $p = \frac{\beta}{1+\beta}$ is obtained.

III. Construction of MDVQ-TLLO

In this section we demonstrate the construction of a practical MDVQ-TLLO quantizer. In contrast to [9] we create a code that does not converge to a repetition code for the combined-description decoder at a high packet-loss rate. To this purpose, we exploit the offsets in the translated-lattice codebooks so that the quantization cell boundaries in one quantizer divide the cells in the second quantizer into smaller regions, each being one quantization cell for the combined-description codebook. Ignoring any overload distortion caused by the codebook translations, the achievable central distortion is reduced without introducing additional side distortions.

For demonstrating the construction, we use dimension two and a $\mathbb{Z}_2$ lattice. While $\mathbb{Z}$ lattices are suboptimal in terms of the normalized second order moment of inertia, their simple geometrical structure of the lattice facilitates the demonstration of the construction. It should be straightforward to extend the method to higher dimensions or other lattices. In this paper, we choose the local codebook size to be $N = 12$.

A. Notation

We assume the first single-description codebook to have structure $\Lambda_1 = \mathbb{Z}_2$ with identity generating matrix: $\mathbf{G} = \mathbf{I}$. The second single-description codebook Λ_2 is a translated version of Λ_1 with offset $\mathbf{o} = [\frac{1}{2} \quad \frac{1}{2}]^T$. We neglect a scaling factor needed in a practical entropy coded system, as we only consider the normalized high-rate distortion.

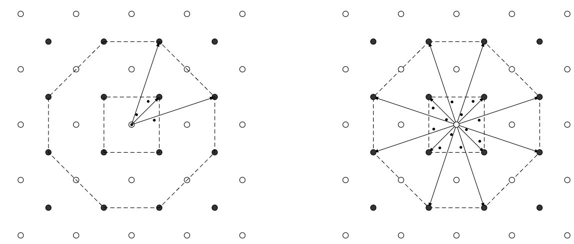

Fig. 2. The white circles represent the Λ_1 lattice and the black circles represent Λ_2 with offset vector $[\frac{1}{2} \quad \frac{1}{2}]^T$. The local codebook $\mathcal{C}'$ is represented in dots inside the Voronoi region of λ^0. The first two shells of the translated $\mathbb{Z}_2$ lattice are connected using dashed lines. (a) shows the codevectors and edges to be stored in the codebooks and (b) shows the complete set of codevectors and edges generated using the set of unitary transforms in Γ (16).

Our two-description quantizer is based on a *graph* with nodes in $\{\Lambda_1 \bigcup \Lambda_2\}$, connecting codevectors of the first and second descriptions. Pairs of lattice vectors in Λ_1 and Λ_2 are *edges* in the graph, denoted as $[\lambda_1, \lambda_2]$ for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. We also define a shift operator for translating edges for an offset vector $s \in \Lambda_1$:

$$[\lambda_1, \lambda_2] + s = [\lambda_1 + s, \lambda_2 + s]. \quad (12)$$

B. Construction of ℓ

The injective MD mapping function ℓ maps each codevector in the combined-description codebook $\mathcal{C}$ to an edge. Due to the codebook structure (3), the mapping function needs to be defined for the local codebook $\mathcal{C}'$ only. The mapping can be extended to the entire vector space by repeating the local mapping using the shift property:

$$\ell(c + s) = \ell(c) + s, \quad (13)$$

for any $c \in \mathcal{C}'$ and $s \in \Lambda$.

The optimal MD mapping ℓ is obtained using the linear optimization approach described in [7]. From the lattice point at the origin, the N closest points in Λ_1 are selected to form an initial set of edges. Figure 2b shows the set of $N = 12$ initial edges. For each possible mapping of a codevector in $\mathcal{C}'$ to an edge, the edge is translated using the shift property to minimize the side distortions. As each possible mapping of codevectors in $\mathcal{C}'$ to edges is associated with an average squared error, the optimal mapping can be obtained using standard linear optimization methods. The mappings of a few selected codevectors in $\mathcal{C}'$ are highlighted in figure 5 by connecting the codevectors to the end points of the corresponding edges using dotted lines.

C. Local codebook $\mathcal{C}'$

The local codebook is trained for a given erasure probability p, equivalently β, to minimize the average total distortion $\bar{D}$ according to (10).

For fixed $\{\ell, \mathcal{C}', \beta\}$, the optimal i'th quantization cell of the encoder $\mathcal{E}$ is given by:

$$\begin{aligned}\mathcal{V}_i \;=\; & \{\, \mathbf{x} : d(\mathbf{x}, \mathbf{c}_i) + \beta d(\mathbf{x}, \ell_1(\mathbf{c}_i)) + \beta d(\mathbf{x}, \ell_2(\mathbf{c}_i)) \\ & \leq d(\mathbf{x}, \mathbf{c}_j) + \beta d(\mathbf{x}, \ell_1(\mathbf{c}_j)) + \beta d(\mathbf{x}, \ell_2(\mathbf{c}_j)) \\ & \forall \;\; j \neq i\}. \end{aligned} \quad (14)$$

For fixed $\{\ell, \mathcal{V}(\mathcal{C}'), \beta\}$, the i'th entry of the optimal local codebook $\mathcal{C}'$ is given by:

$$\mathbf{c}_i \;=\; \arg\min_{\mathbf{y} \in \mathbb{R}^k} \mathrm{E}[d(\mathbf{x}, \mathbf{y}) | \mathbf{x} \in \mathcal{V}_i]. \quad (15)$$

Using the optimal encoder (14) and decoder (15) criteria, the optimal local codebook $\mathcal{C}'$ is obtained by iteratively optimizing the encoder and the decoder until convergence.

To encode an input vector x, the closest lattice point $\lambda \in \Lambda_1$ is found first. An extended local codebook $\mathcal{C}''$ is then defined as $\lambda + \mathcal{C}'$ plus a few codevectors from neighboring lattice cells sitting on the boundaries. The codevector $\mathbf{c}$ in the extended local codebook $\mathcal{C}''$ that minimizes the weighted distortion is selected. The mapping function ℓ is used to obtain the descriptions for $\mathbf{c}$, one description in Λ_1 and the other in Λ_2.

D. Complexity reduction

As shown in figure 2, the selected initial edges consist of translated lattice points with equal distance to the origin within different shells. This structure can be used to greatly reduce the training complexity and storage of the local codebook. For a set Γ of carefully chosen unitary transform matrices, we only need to train and store a subset of the local codebook as well as the corresponding edges. The complete set can be generated using the subset and transforms in Γ.

Unlike the transformation group defined in [7], Γ must preserve both the lattice Λ_1 and the translated lattice Λ_2. For the $\mathbb{Z}_2$ lattice and a translation of $[\frac{1}{2} \;\; \frac{1}{2}]^T$, Γ is selected to be:

$$\Gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}. \quad (16)$$

It is trivial to show that all $\gamma \in \Gamma$ are unitary transforms, and the requirements of $\gamma\lambda_1 \in \Lambda_1, \forall \lambda_1 \in \Lambda_1$ and $\gamma\lambda_2 \in \Lambda_2, \forall \lambda_2 \in \Lambda_2$ are fullfilled. The concept is illustrated in figure 2a, where only a subset of the edges from the first two shells of closest translated-lattice points are shown. Using Γ of (16), the complete set of edges in figure 2b can be generated from the subset.

IV. Experiments

The proposed method has been implemented using the 12 edges from the first two shells of the translated lattice Λ_2 shown in figure 2. Local codebooks were trained for $p = 0 \ldots 0.3$ using 10^6 uniformly distributed random vectors generated within the Voronoi cell of the lattice vector at the origin. The quantization cells of the combined-description codebooks are shown in figure 5 for selected p. The codevectors inside the dashed region form the local codebook, and the structure enforced by (3) and (16) are clearly visible. The mapping ℓ for the selected subset of the local codebook is shown in dotted

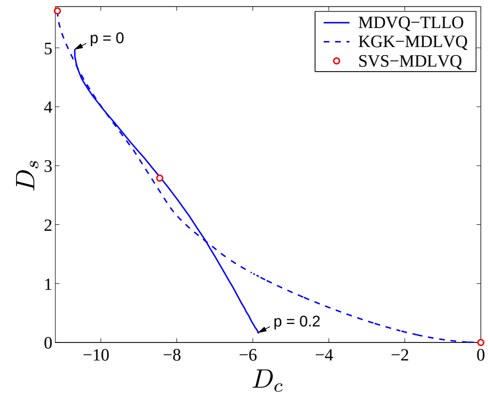

Fig. 3. Performance comparison of normalized central and side distortions in dB. A few channel optimized operating points of MDVQ-TLLO are highlighted.

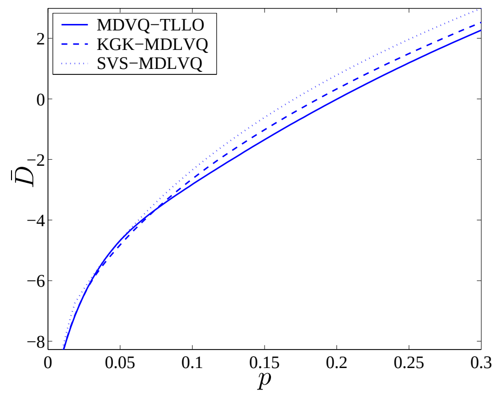

Fig. 4. Performance comparison of the average total distortion in dB using channel optimized quantizers.

lines that connect the codevectors to the end points of the corresponding edges.

As observed in [9], some quantization cells of the local codebook expand while others shrink when p increases. The phenomena can be explained by the unequal distribution of side distortions among the codevectors in the local codebook. As p increases, the codevectors with end points of associated edges far away, causing larger side distortions, will have shrunk quantization cells. At the same time, the codevectors with lower side distortions will have expanded quantization cells. In contrast to [9], the local codebook converges to the interleaved set of the lattice and the translated lattice, resulting in a decreased central distortion.

The central and side distortions were estimated using an additional 10^6 uniformly distributed random vectors. The distortions are normalized such that 0 dB corresponds to the side distortion using the optimal lattice (A_2). The results are compared to the KGK-MDLVQ method in figure 3. Our method is shown to perform close to KGK-MDLVQ at small p. As p increases, KGK-MDLVQ performs better than MDVQ-TLLO. This difference results from usage of the suboptimal $\mathbb{Z}_2$ lattice for the implementation of MDVQ-TLLO whereas the implementation of KGK-MDLVQ uses an optimal hexagonal lattice. However, using codebooks optimized for higher erasure probabilities, MDVQ-TLLO has significantly better performance.

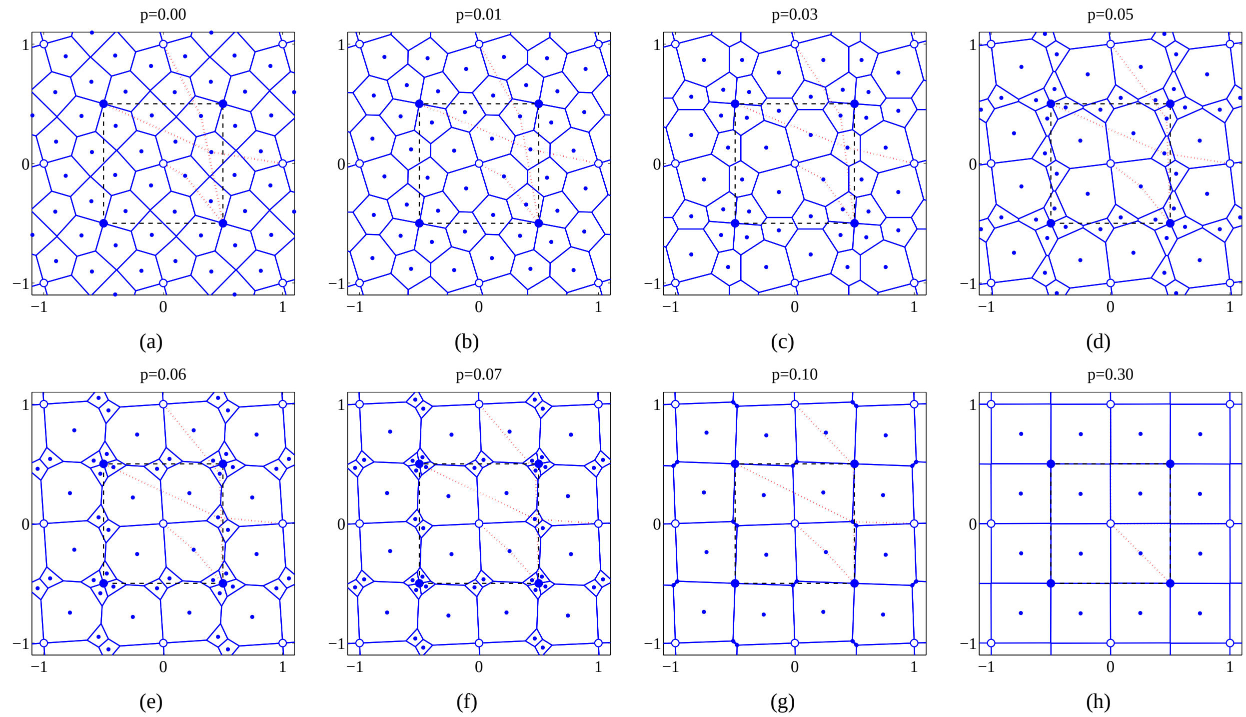

Fig. 5. The quantization cells of the combined-description codebook $\mathcal{C}$. The white circles represent Λ_1 and the black circles represent Λ_2. The Voronoi cell of λ^o is highlighted using dashed lines. The MD mapping ℓ is shown using dotted lines for selected codevectors.

The distortion behavior can be explained from figure 5. At $p = 0$, despite our implementation with the $\mathbb{Z}_2$ lattice for Λ_1 and Λ_2, the local codebook has a structure resembling the optimal lattice codebook of the KGK-MDVLQ implementation. As p increases, the local codebook is dominated by fewer codevectors, and loses the near-optimal cell shape. However, the figure shows that the combined-description codebook of MDVQ-TLL converges to interleaved codevectors from Λ_1 and Λ_2 with increasing p. At high loss rates its central distortion is reduced by 6 dB over the side distortions. This contrasts with KGK-MDLVQ which converges to a repetition code, which has a central distortion that improves 0 dB over the side distortions.

Next we evaluate the average total distortion $\bar{D}$ for different erasure probabilities $0.01 \leq p \leq 0.3$. The optimal operating point of the SVS-MDLVQ method is chosen by the index N that minimizes $\bar{D}$. For KGK-MDLVQ and MDVQ-TLLO, the optimal operating point is selected using the optimized local codebooks. The results are shown in figure 4. As expected, KGK-MDLVQ performs better than SVS-MDLVQ over the entire range of erasure probabilities. For $p > 0.075$, MDVQ-TLLO is shown to perform better than both KGK-MDLVQ and SVS-MDLVQ.

V. Conclusions

This paper shows that translated lattices can be exploited to improve the performance of the central decoder in MDVQ. The method is demonstrated using the two-dimensional $\mathbb{Z}_2$ lattice, and a clear advantage is shown over existing methods [7][9] for high-loss erasure channels. While the method of [9] converges to repetition codes with 0 dB improvement of central distortion over the side distortion, our method converges to interleaved code vectors, with a 6 dB lower central distortion compared to the side distortion. The improvement is gained with insignificant additional computational complexity and storage requirements compared to the approach of [7].

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